ON THE CONCEPT OF POSSIBILITY IN QUANTUM MECHANICS

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Abstract. In this short paper I introduce a concept of *possibility* in order to vindicate Everett's Theory of many worlds. The main idea is that there is only one world: the real. After the wave-collapse, we have again only one world, but many possible worlds which have a different grade of possibility. In this sense *possible* exclude the *reality*, opposing a long modal tradition.

1. Symbolic Notation

Be given the eigen-value problem $\mathcal{O}|\varphi_i\rangle = \lambda_i|\varphi_i\rangle$, where $\mathcal{O}$ is an hermitian operator (a particular linear operator) representing mathematically an observable O. An observable is anything which we can measure. von Neumann spoke of *Grösse*. $|\varphi_i\rangle$ is an eigen-vector, representing the eigen-state in which the wave-function collapses after a measurement. Finally, λ_i is a real number: the measure of O. I will write the wave-collpase this way:

$$m_i(\mathcal{O}) \to |\varphi_i\rangle \vDash m_i(\mathcal{O}) = \lambda_i \tag{1}$$

I.e. the measurement m_i on $\mathcal{O}$ collapses the wave-function of $\mathcal{O}$ in one of its eigen-state $|\varphi_i\rangle$ in which is it true that $m_i(\mathcal{O}) = \lambda_i$, it is true that $\mathcal{O}$ measures just λ_i. I would insist on this notation. The arrow $\to$ suggests an implication from the measurement to the collapse. Obviously, this arrow is a meta-linguistic sign. The formulation of the conseguent suggests a parallel with Model Theory. The eigen-vector is a sort of model in which a formula is true or false (in this case, true). In fact, we can speak of measurement only in the context of an eigen-state.

One could observe that we neglected the amplitude of probability associated to a *possible* eigen-state. As you can see below, we can express that a measurement implies a collapse with a propability $P = r_i$ in the following manner:

$$m_i(\mathcal{O}) \xrightarrow{P=r_i} |\varphi_i\rangle \vDash m_i(\mathcal{O}) = \lambda_i \tag{2}$$

2. Modal Definitions

Let us state that $|\varphi_{\mathcal{O}}\rangle = c_i|\varphi_i\rangle + c_j|\varphi_j\rangle$ and that $m_i(\mathcal{O}) \to |\varphi_i\rangle \vDash m_i(\mathcal{O}) = \lambda_i$;[1], then $|\varphi_j\rangle$ is **possible** as regards $|\varphi_i\rangle$; on the contrary, if were $m_j(\mathcal{O}) \to |\varphi_j\rangle \vDash m_j(\mathcal{O}) = \lambda_j$, $|\varphi_i\rangle$ would be **possible** as regards $|\varphi_j\rangle$. *Possibility* is no more in this context a predicate, but a *relation amongs eigen-states*. Symbolically:

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[1] I.e. the measurement m_i on an observable O collapses the wave-function $|\varphi_{\mathcal{O}}\rangle$ in the eigen-state $|\varphi_i\rangle$ in which the eigen-value of the operator associated to O equals λ_i. In other words, it is true in the eigen-state $|\varphi_i\rangle$ that the eigen-function $m_i(\mathcal{O})$ has the value λ_i.

Definition 1. *(Relative Possibility) If* $|\varphi_{\mathcal{O}}\rangle = c_i|\varphi_i\rangle + c_j|\varphi_j\rangle$ *and* $m_i(\mathcal{O}) \to |\varphi_i\rangle \vDash m_i(\mathcal{O}) = \lambda_i$*, then* $|\varphi_j\rangle \sim |\varphi_i\rangle$.[2]

In other terms: $|\varphi_j\rangle$ is possible for $|\varphi_i\rangle$.
$|\varphi_i\rangle$ is *possible tout-court* when it exists a real eigen-state of which $|\varphi_i\rangle$ is possible (see below **Definition 2**). If we have that $m_j(\mathcal{O}) \to |\varphi_j\rangle \vDash m_j(\mathcal{O}) = \lambda_j$, then $|\varphi_i\rangle$ is possible as regards $|\varphi_j\rangle$. Nevertheless, it could be another measurement m_i such that $m_i(\mathcal{O}) \to |\varphi_i\rangle \vDash m_i(\mathcal{O}) = \lambda_i$. In this situation, $|\varphi_i\rangle$ is *real* and *no* more possible. But we had not required that $|\varphi_i\rangle$ be possible as regards *any* eigen-state, but only that there be *at least* an eigen-state, of which $|\varphi_i\rangle$ were possible. Of course, if $|\varphi_i\rangle$ is possible as regards $|\varphi_j\rangle$, then it is possible tout-court (symbolically, $\sim |\varphi_i\rangle$[3]).

Definition 2. *(Possible)* $\sim |\varphi_i\rangle \overset{def}{=} \exists|\varphi_j\rangle$ *such that* $|\varphi_i\rangle \sim |\varphi_j\rangle$.

$\sim$, understood as a relation and not as an operator, is anti-simmetrical, transitive, but *not* reflexive. According our definition, the possibility excludes the reality. In fact, only the braches originating with the measurement which don't become real are possible.

$$\neg(|\varphi_i\rangle \sim |\varphi_i\rangle) \qquad \text{(Anti-Simmetry)} \tag{3}$$

$$|\varphi_i\rangle \sim |\varphi_j\rangle \to \neg(|\varphi_j\rangle \sim |\varphi_i\rangle) \qquad \text{(Not Reflexivity)} \tag{4}$$

$$(|\varphi_i\rangle \sim |\varphi_j\rangle \wedge |\varphi_j\rangle \sim |\varphi_k\rangle) \to |\varphi_i\rangle \sim |\varphi_k\rangle \qquad \text{(Transitivity)} \tag{5}$$

Definition 3. *(Determinism) If* $|\emptyset\rangle \sim |\varphi_i\rangle$*, then* $|\varphi_i\rangle$ *is a deterministic eigen-state (*$\mathsf{Det}(|\varphi_i\rangle)$*) and* m_i *a deterministic process.*

In other words, if there are no possibilities apart a unique eigen-state, then this eigen-state is deterministic. There is no choice to make. $|\varphi_i\rangle$ must realize itself. On the contrary;

Definition 4. *(Absurdity) If* $|\varphi_i\rangle \sim |1\rangle$*, then* $|\varphi_i\rangle$ *is an absurd eigen-state.*

3. Shaping the Universe of Possible Worlds

Until now, we saw that possibility excludes reality. So we can imagine a world splitted up in two distinct parts, possible and real, with a sharp limit between them. This way, the realm of possible can sound caothic. Any non-real branch is on the same ontological level than another non-real branch. I think that we can distinguish many grades of possibility. There is a possibility near the real, and there is a possibility too far from our sight. The fact that in next presidentials Obama could be not re-elected is a possibility; also the existence of a fancy figure is a possibility. But we reckon that we are facing two types of possibilities, or that the first possibility is more possible that the second.
So what we need is a map where drawing any branch arising from the wave collapse. The idea is to assign a length to a branch. A branch longer than another, is so more possible. How assigning a length? The length of a branch is its probability. So if the branch b_i has a probability $1/3$ and the branch b_j a probability $1/2$, then the second branch is more possibile (nearer the reality) than the first.

[2]We use the symbol $\sim$, following C.I. Lewis to indicate the possibility.
[3]I hope that using the same symbol denoting both a relation and a predicate be not confusing.

Definition 5. *(Comparison amongs Diverse Possibilities) Be* $|\varphi_{\mathcal{O}}\rangle = c_i|\varphi_i\rangle + c_j|\varphi_j\rangle + c_k|\varphi_k\rangle$ *and* $m_i(\mathcal{O}) \to |\varphi_i\rangle \vDash m_i(\mathcal{O}) = \lambda_i$*, then if* $m_i(O) \xrightarrow{P=|c_j|^2} |\varphi_j\rangle \vDash m_i(\mathcal{O}) = \lambda_j$[4] *and* $m_i(\mathcal{O}) \xrightarrow{P=|c_k|^2} |\varphi_k\rangle \vDash m_i(\mathcal{O}) = \lambda_k$*, with* $c_j \leq c_k$ *and* $c_j, c_k \in \mathbb{C}$*, then* $|\varphi_k\rangle$ *is more possible as regards* $|\varphi_i\rangle$ *than* $|\varphi_j\rangle$*; where* $|\varphi_j\rangle, |\varphi_k\rangle$ *are not realized eigen-states and* $|\varphi_i\rangle$ *the eigen-state caused by* m_i*.*

Example: $|\varphi_{\mathcal{O}}\rangle = c_1|\varphi_1\rangle + c_2|\varphi_2\rangle + c_3|\varphi_3\rangle + \ldots + c_n|\varphi_n\rangle$, but only $|\varphi_3\rangle$ becomes the eigen-state in which $m_i(\mathcal{O}) = \lambda_3$, then, as regards $|\varphi_3\rangle$ all the $|\varphi_1\rangle, |\varphi_2\rangle, |\varphi_4\rangle, |\varphi_5\rangle, \ldots, |\varphi_n\rangle$ are only possible, with diverse grade of possibility.

Definition 6. *(Relation amongs Possibilities) Be* $|\varphi_{\mathcal{O}}\rangle = c_1|\varphi_1\rangle + \ldots + c_n|\varphi_n\rangle$, $c_1, c_n \in \mathbb{C}$*; if* $m_i(\mathcal{O}) \to |\varphi_i\rangle \vDash m_i(\mathcal{O}) = \lambda_i$*, then* $\forall j \neq i, |\varphi_j\rangle \sim |\varphi_i\rangle$ *and given two eigen-states whatever* $|\varphi_j\rangle$ *and* $|\varphi_k\rangle$ *such that* $|c_j|^2 \geq |c_k|^2$*, then* $(|\varphi_j\rangle \sim^{\pm} |\varphi_k\rangle) \sim |\varphi_i\rangle$*. I.e.,* $|\varphi_j\rangle$ *is more possible for* $|\varphi_i\rangle$ *than* $|\varphi_k\rangle$*.*

As we have stressed, an eigen-state is possible only as regards a real eigen-state. In the following, we omit the reference to the real eigen-state, if not necessary to understand. We may say, therefore, that the relation $\sim^{\pm}$ orders the eigen-states of a wave-function according to their possibility as regards a $|\varphi_i\rangle$.

Definition 7. *(Transitivity)* $(|\varphi_j\rangle \sim^{\pm} |\varphi_i\rangle \wedge |\varphi_i\rangle \sim^{\pm} |\varphi_k\rangle) \to |\varphi_j\rangle \sim^{\pm} |\varphi_i\rangle$

Definition 8. *($|\varphi_i\rangle$ and $|\varphi_k\rangle$ have the Same Grade of Possibility)* $(|\varphi_i\rangle \sim^{\pm} |\varphi_k\rangle \wedge |\varphi_k\rangle \sim^{\pm} |\varphi_i\rangle) \to |\varphi_k\rangle \sim^{=} |\varphi_i\rangle$

I.e. if one eigen-state is more possible than another, and this last is more possible than the first, then the two eigen-state have the same probability.

Definition 9. *(Simmetry)* $|\varphi_i\rangle \sim^{=} |\varphi_i\rangle$

Definition 10. *(Lower bound) It exists an eigen-state* $|\varphi_j\rangle$ *such that* $\forall |\varphi_i\rangle, |\varphi_i\rangle \sim^{\pm} |\varphi_j\rangle$*.*[5]

Definition 11. *(Upper bound) It exists an eigen-state* $|\varphi_j\rangle$ *such that* $\forall |\varphi_i\rangle, |\varphi_j\rangle \sim^{\pm} |\varphi_i\rangle$*.*[6]

Absurdity is the relation between a possible event with $P = 0$[7] and a deterministic eigen-state with $P = 1$. For example, be $|\varphi_{\mathcal{O}}\rangle = c_1|\varphi_1\rangle + c_2|\emptyset\rangle$, then $|\varphi_{\mathcal{O}}\rangle$ can collapse only on $|\varphi_1\rangle$; therefore, $|\emptyset\rangle$ is absurd as regards $|\varphi_1\rangle$. In the same situation, $|\varphi_1\rangle$ is deterministic as regards $|\emptyset\rangle$.

Definition 12. *(Impossible) An eigen-state* $|\varphi_i\rangle$ *is* impossible *(not in a relation of absurdity), when* $\forall |\varphi_{\mathcal{O}}\rangle = c_1|\varphi_1\rangle + \ldots + c_n|\varphi_n\rangle$ *and every eigen-function* m_i $P(|\varphi_i\rangle) = 0$ [i.e. $\neg\Diamond|\varphi_i\rangle$].

In other words, an eigen-state is impossible when it doesn't exist a wave-function of which it is a possible eigen-state.

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[4]As noted above $m_i(\mathcal{O}) \xrightarrow{P=|c_j|^2} |\varphi_j\rangle \vDash m_i(\mathcal{O}) = \lambda_j$ means that the measurement m_i on O has a value λ_j with a probability $|c_j|^2$. I.e., m_i implies with a probability $P = |c_j|^2$ a collapse in the eigen-state $|\varphi_j\rangle$ in which the value of the eigen-function $m_i(\mathcal{O})$ equals λ_j.

[5]The *event* $|\varphi_j\rangle$ is the less probable event (always in reference to a real event). **NB:** $|\varphi_j\rangle$ is the less possible event but not for that *impossible*.

[6]$|\varphi_j\rangle$ is the eigen-state most possible as regards the real one, but it is **not** *necessary*.

[7]I.e. with probability equal to 0.

Definition 13. *(Necessary) On the contrary, an eigen-state $|\varphi_i\rangle$ is* necessary *(not in a relation of determinism) when $\forall|\varphi_{\mathcal{O}}\rangle = c_1|\varphi_1\rangle + \ldots + c_n|\varphi_n\rangle$ and every measurement $P(|\varphi_i\rangle) = 1$ $[\Box|\varphi_i\rangle]$.*[8]

We can re-phrase the two previous definitions as follows:

Definition 14. *$|\varphi_i\rangle$ is impossible when it is in a relation of absurdity in every context $[\neg\Diamond|\varphi_i\rangle]$.*[9]

Definition 15. *$|\varphi_i\rangle$ is necessary when it is deterministic in every context $[\Box|\varphi_i\rangle]$.*[10]

We can explain the difference between *being deterministic* and *being necessary* with an example: $|\varphi_i\rangle$ is in a relation of determinism with an eigen-state $|\varphi_j\rangle$ when the square of the module of the weight associated to $|\varphi_i\rangle$, $(|c_i|^2)$ equals 1. So, $|\varphi_i\rangle$ is not always deterministic, but only when its probability is 1. When it is *always* deterministic, then it is necessary.
Classical Physics is deterministic; therefore, the raining today is a deterministic fact, but not necessary. It could not rain. It would be sufficient that the initial conditions were different. In this case, it is deterministic, according to the present situation, that it rains, but it is not necessary. It exists obviously at least a situation in which it doesn't rain. For example in the desert.
On the contrary, if it rains despite any possible context, then raining is really necessary. A similar argument obtains for the impossibility, inasmuch *necessity* and *impossibility* are related together by definition:

Definition 16. *$\Box A \stackrel{def}{=} \neg \sim \neg A$.*

4. The Branching after the Wave-Collpase

Let us state that it exists a wave-function $|\varphi_{\mathcal{O}_i}\rangle = c_1|\varphi_1\rangle + \ldots + c_n|\varphi_n\rangle$. The measurement collapses the state in which the system lays in an eigen-state $|\varphi_i\rangle$, as regards to which any other eigen-state is only possible. We make a choice among these possible eigen-state, taking a $|\varphi_j\rangle$.
Now, we measure the observable O_j and its wave-function collapses in the eigen-state $|\varphi_j\rangle$. We choose amongs the possible states for $|\varphi_j\rangle$ a $|\varphi_k\rangle \neq |\varphi_i\rangle$. As our relation of possibility is transitive, $|\varphi_k\rangle$ will be possible by $|\varphi_j\rangle$ for $|\varphi_i\rangle$ in our first measurement. But what is the difference between $|\varphi_j\rangle$ and $|\varphi_k\rangle$ as regards $|\varphi_i\rangle$ in our first measurement?
Intuitively, $|\varphi_k\rangle$ is more far than $|\varphi_j\rangle$ from $|\varphi_i\rangle$. A way to compute the distance could be to use again the probability, compounding the probability of $|\varphi_j\rangle$ for $|\varphi_i\rangle$ with that of $|\varphi_k\rangle$ for $|\varphi_i\rangle$, as in an oriented graph. The vertices are the states or the eigen-states (real or possible) and the arrows their probabilities.
In this case, we have $|\varphi_i\rangle \xrightarrow{R} |\varphi_j\rangle$ and $|\varphi_j\rangle \xrightarrow{S} |\varphi_k\rangle$. Then, the distance between the vertices $|\varphi_i\rangle$ and $|\varphi_k\rangle$ equals $S \circ R$: $|\varphi_i\rangle \xrightarrow{S\circ R = T} |\varphi_k\rangle$. Now, the distance of $|\varphi_k\rangle$ from $|\varphi_i\rangle$ is $P(|\varphi_k\rangle||\varphi_j\rangle)$, i.e. the conditioned probability of $|\varphi_k\rangle$, given $|\varphi_j\rangle$.

[8] $|\varphi_i\rangle$ is not only deterministic, but deterministic in a context whatever.

[9] For example, $|\varphi_i\rangle$ can be absurd in a given situation, but it can be not only possible, but also real in another situation. It is impossible when it is absurde in any context.

[10] In fact, $|\varphi_i\rangle$ can be deterministic in a given context and be only possible in another context. When it is deterministic always, despite any particular situation, then it is necessary.

5. Note

Finally, the set of the probabilities associated to a wave-function $|\varphi_{\mathcal{O}}\rangle, \mathcal{P}_{|\varphi_{\mathcal{O}}\rangle} = \{|c_1|^2, |c_2|^2, \ldots, |c_n|^2\}$ is well ordered by the relation $\sim^{\pm}$. This well ordered set $\mathcal{P}^{\sim^{\pm}}_{|\varphi_{\mathcal{O}}\rangle}$ has an *infimum* (see above the *lower bound*), a *supremum* (see above the *upper bound*) and any couple of probabilities in it has a min and a sup. So, $\mathcal{P}^{\sim^{\pm}}_{|\varphi_{\mathcal{O}}\rangle}$ is a lattice. Furthermore, because $P(\neg|\varphi_i\rangle) = 1 - P(|\varphi_i\rangle)$, $\mathcal{P}^{\sim^{\pm}}_{|\varphi_{\mathcal{O}}\rangle}$ is orto-complemented. Finally, it is *distributive*. Also it is a Boolean Algebra: $\mathfrak{O} =< |c_i|^2_{i\in\mathbb{N}}, \neg, +, 0, 1 >$.

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